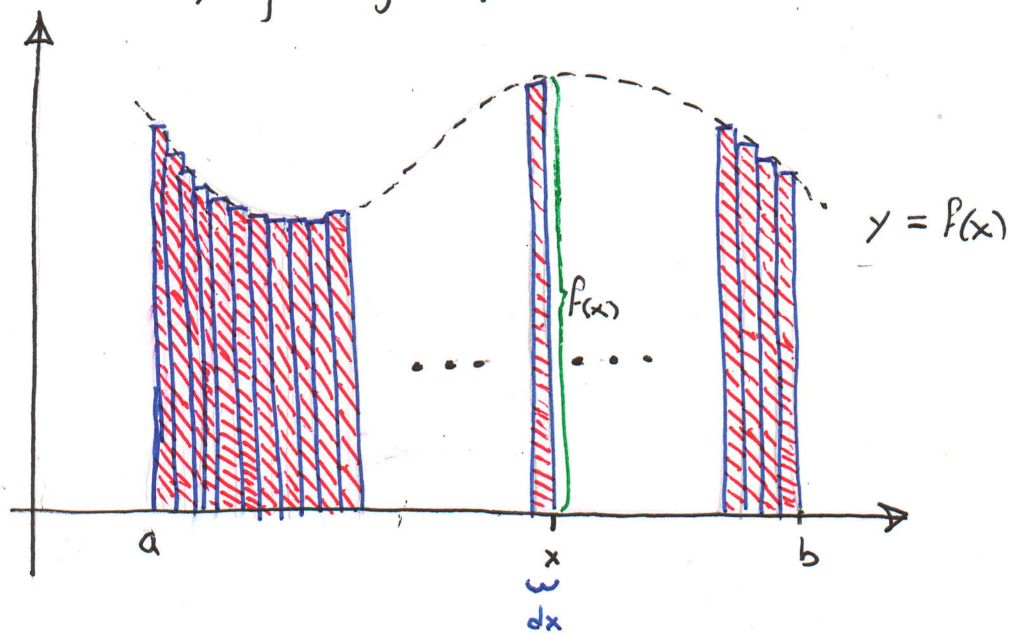
Area and Riemann Sum Lecture 3

In tackling the area under the curve problem, we have agreed to view the curve $y = f(x)$ as the outline traced by the tips of thickly packed narrow rectangular columns. (I tend to think of these columns as matches of identical thickness and varying lengths).



$$\int_a^b f(x)\,dx \equiv \text{sum areas of}$$

all thin rectangular columns from $x = a$ to $x = b$.

To carry out the summation of the areas of thin rectangles we have to be methodical!

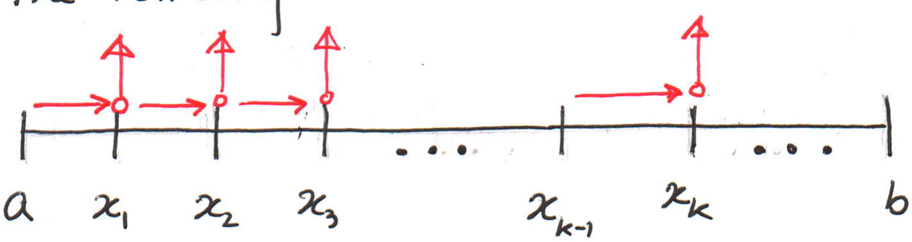
For every integer $n \in \mathbb{N}$ $(n \geq 1)$. Define $R_n = \sum_{k=1}^{n} f(x_k)\Delta x_n$

as follows:

$\Delta x_n = \frac{b-a}{n}$ – base $\underset{a \qquad\qquad b}{\vdash\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!\dashv}$ divided into n equal line segments. These will serve as fundaments to our rectangular columns.

The height of each rectangle will be determined at the rightmost point of the interval segment on which it (the rectangle) stands. (R_n for right sum)

Beginning construction from a and moving to the right we get the following scheme or construction blueprint.



$x_1 = a + 1 \cdot \Delta x_n = a + 1 \cdot \frac{b-a}{n}$ – from a displace by 1 block.

$x_2 = a + 2\Delta x_n = a + 2 \cdot \frac{b-a}{n}$ – from a displace by 2 blocks

$x_3 = a + 3\Delta x_n = a + 3\frac{b-a}{n}$ – from a displace by 3 blocks.

and in general

$x_k = a + k\Delta x_n = a + k\Delta x_n = a + k\frac{b-a}{n}$ - To get to the k^{th} point, move from a k blocks to the right.

I tend to think of the interval $[a,b]$ as a kind of 1 dimensional Manhattan, Δx_n as the city blocks, x_k as construction site on the k^{th} block of the k^{th} skyscraper, and $f(x_k)$ as its height.

R_n means that n skyscrapers were built and the construction site was the rightmost point on each block.

Clearly $R_n = \sum_{k=1}^{n} f(x_k)\Delta x_n = \sum_{k=1}^{n} f\left(a+k\frac{b-a}{n}\right)\frac{b-a}{n}$

If n is very large, the rectangles are very thin and fit along the curve snuggly.

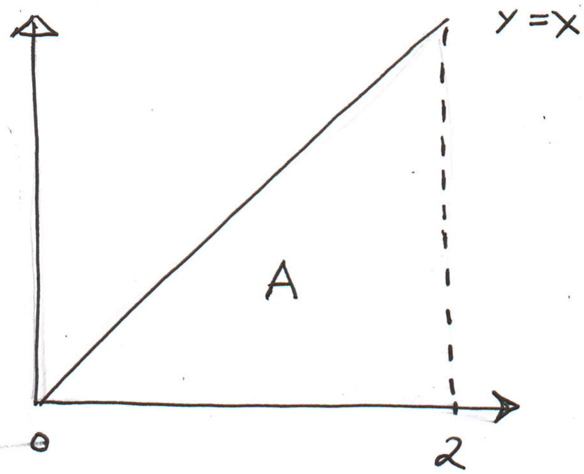
We expect therefore that

$$\boxed{\int_a^b f(x)\,dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{k=1}^{n} f\left(a+k\frac{b-a}{n}\right)\frac{b-a}{n}}$$

Let us put this idea to the test.

<u>Ex.</u> Calculate $\int_0^2 x\,dx$

<u>Solution:</u>



This is just a right triangle with width 2 and height 2. Thus the area is $\frac{1}{2}\cdot 2\cdot 2 = 2$.

Hence we expect $\int_0^2 x\,dx = 2$.

Indeed $\Delta x_n = \frac{2}{n}$, $x_k = 0 + k\,\Delta x_n = k\frac{2}{n}$,

and $f(x_k) = f(k\frac{2}{n}) = k\frac{2}{n}$ because $f(x) = x$.

Hence $R_n = \sum_{k=1}^{n} f(x_k)\,\Delta x_n = \sum_{k=1}^{n} \left(k\frac{2}{n}\right)\frac{2}{n} =$

$$= \left(\frac{2}{n}\right)^2 \sum_{k=1}^{n} k = \left(\frac{2}{n}\right)^2 \frac{n(n+1)}{2}$$

$$\int_0^2 x\,dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \left(\frac{2}{n}\right)^2 \frac{n(n+1)}{2} = \frac{2^2}{2} = 2.$$

It works!

Ex. Calculate $\int_1^4 (2+3x)\,dx$

Solution: $f(x) = 2+3x$, $\Delta x_n = \frac{4-1}{n} = \frac{3}{n}$

$x_k = 1 + k\Delta x_n = 1 + k\frac{3}{n}$, $f(x_k) = 2 + 3x_k =$

$$= 2 + 3\left(1 + k\tfrac{3}{n}\right) = 5 + 3\left(k\tfrac{3}{n}\right)$$

so $$R_n = \sum_{k=1}^{n}\left[5 + 3\left(k\tfrac{3}{n}\right)\right]\frac{3}{n} = \sum_{k=1}^{n} 5\frac{3}{n} + \sum_{k=1}^{n} 3\left(k\tfrac{3}{n}\right)\frac{3}{n}$$

$$= 5\cdot\frac{3}{n}\sum_{k=1}^{n} 1 + 3\left(\frac{3}{n}\right)^2 \sum_{k=1}^{n} k$$

$$= 5\cdot\frac{3}{n}\cdot n + 3\left(\frac{3}{n}\right)^2 \frac{n(n+1)}{2}$$

$$\int_1^4 (2+3x)\,dx = \lim_{n\to\infty}\left[5\cdot\frac{3}{n}\cdot n + 3\left(\frac{3}{n}\right)^2 \frac{n(n+1)}{2}\right]$$

$$= 5\cdot 3 + 3^3\cdot\frac{1}{2} = \frac{30}{2} + \frac{27}{2} = \frac{57}{2}.$$

Ex. Calculate $\int_0^2 x^2 dx$

Solution: This is no longe trivial geometrically.

$f(x) = x^2$; $\Delta x_n = \frac{2}{n}$; $x_k = 0 + k\Delta x_n = k\frac{2}{n}$

$f(x_k) = x_k^2 = (k\frac{2}{n})^2$

Thus $R_n = \sum_{k=1}^{n} f(x_k)\Delta x_n = \sum_{k=1}^{n} (k\frac{2}{n})^2 \frac{2}{n}$

$= \left(\frac{2}{n}\right)^3 \sum_{k=1}^{n} k^2 = \left(\frac{2}{n}\right)^3 \frac{n(n+1)(2n+1)}{6}$

Thus $\lim_{n\to\infty} R_n = \lim_{n\to\infty} \left(\frac{2}{n}\right)^3 \frac{n(n+1)(2n+1)}{6}$

$= 2^3 \cdot \frac{2}{6} = \frac{2^3}{3} = \frac{8}{3}$.

Ex. Calculate $\int_0^1 (5 - 3x^2) dx$

Solution:

$f(x) = 5 - 3x^2$; $\Delta x_n = \frac{1}{n}$; $x_k = 0 + k\frac{1}{n}$

$R_n = \sum_{k=1}^{n} f(x_k)\Delta x_n = \sum_{k=1}^{n} \left[5 - 3(k\frac{1}{n})^2\right]\frac{1}{n}$

$= \sum_{k=1}^{n} 5\frac{1}{n} - 3\left(\frac{1}{n}\right)^3 \sum_{k=1}^{n} k^2 = 5\frac{1}{n}\cdot n - 3\left(\frac{1}{n}\right)^3 \frac{n(n+1)(2n+1)}{6}$

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty}\left[5\cdot\tfrac{1}{n}\cdot n - 3\left(\tfrac{1}{n}\right)^3 \frac{n(n+1)(2n+1)}{6}\right]$$

$$= 5 - 3\cdot\tfrac{2}{6} = 5-1 = 4.$$

<u>Ex.</u> $\int_1^2 (2-2x+6x^2)\,dx = ?$

<u>Solution:</u>

$f(x) = 2-2x+6x^2$; $\Delta x_n = \frac{1}{n}$; $x_k = 1 + k\frac{1}{n}$

$$f(x_k) = 2-2x_k+6x_k^2 = 2-2\left(1+k\tfrac{1}{n}\right)+6\left(1+k\tfrac{1}{n}\right)^2$$

$$= 2-2-2\tfrac{1}{n}k + 6\left(1+2\tfrac{1}{n}k + \left(\tfrac{1}{n}\right)^2k^2\right)$$

$$= 6 + 10\tfrac{1}{n}k + 6\left(\tfrac{1}{n}\right)^2k^2$$

$$R_n = \sum_{k=1}^{n} f(x_k)\Delta x_n = \sum_{k=1}^{n}\left(6+10\tfrac{1}{n}k + 6\left(\tfrac{1}{n}\right)^2k^2\right)\tfrac{1}{n}$$

$$= \sum_{k=1}^{n} 6\tfrac{1}{n} + \sum_{k=1}^{n} 10\left(\tfrac{1}{n}\right)^2 k + \sum_{k=1}^{n} 6\left(\tfrac{1}{n}\right)^3 k^2$$

$$= 6\tfrac{1}{n}\cdot n + 10\left(\tfrac{1}{n}\right)^2\frac{n(n+1)}{2} + 6\left(\tfrac{1}{n}\right)^3\frac{n(n+1)(2n+1)}{6}$$

$$\lim_{n\to\infty} R_n = 6 + \tfrac{10}{2} + 6\cdot\tfrac{2}{6} = 6+5+2 = 13.$$

Ex. Compute $\int_0^3 x^3\,dx$.

Solution:

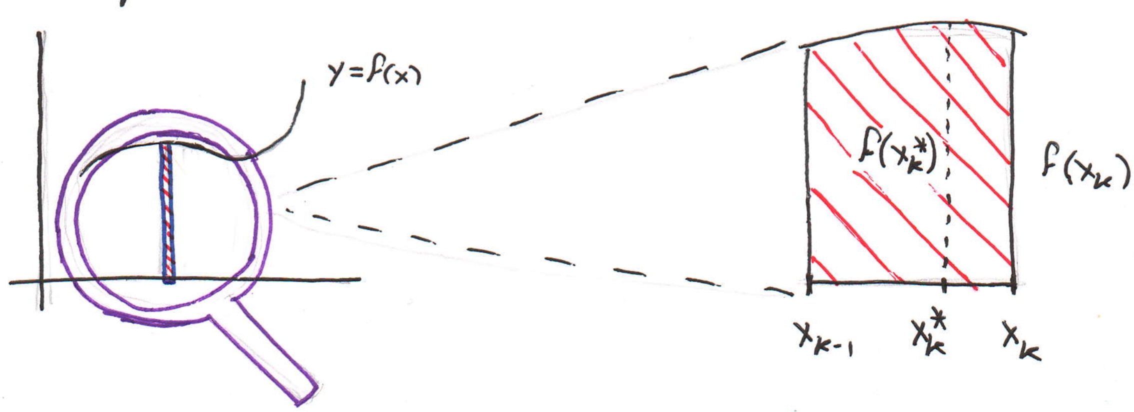
$$R_n = \sum_{k=1}^{n} \left(k\frac{3}{n}\right)^3 \frac{3}{n} = \left(\frac{3}{n}\right)^4 \sum_{k=1}^{n} k^3$$

$$= \left(\frac{3}{n}\right)^4 \left[\frac{n(n+1)}{2}\right]^2$$

$$\lim_{n\to\infty} R_n = \frac{3^4}{2^2} = \frac{81}{4}$$

Thus $\int_0^3 x^3\,dx = \frac{81}{4}$.

When $f(x)$ is continuous and $[x_{k-1}, x_k]$ is a very short interval any point $x_k^* \in [x_{k-1}, x_k]$ yields approximately the same altitude $f(x_k^*)$ as the one constructed at the rightmost point of the interval. That is $f(x_k^*) \approx f(x_k)$



Thus we expect

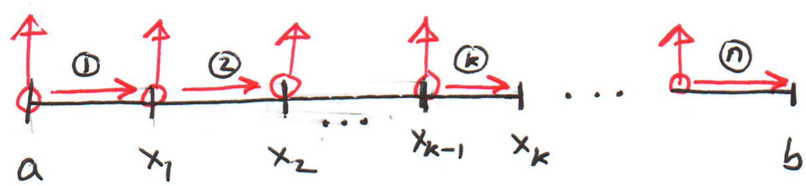
$$\int_a^b f(x)\,dx = \lim_{n\to\infty} G_n = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x_n$$

where G_n stands for general point in the k^{th} interval, x_k^* is used to compute Riemann sum.

Ex. Express $\int_a^b f(x)\,dx$ as $\lim_{n\to\infty} L_n$

where L_n means that we construct the altitude $f(x)$ at the leftmost point of each interval segment.

Solution:



To get to the k^{th} construction point from a simply displace $(k-1)$ intervals. Thus $x_k = a + (k-1)\Delta x_n$

$= a + (k-1)\frac{b-a}{n}$ for $1 \le k \le n$.

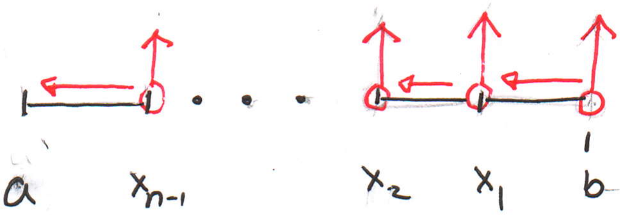
Hence $L_n = \sum_{k=1}^{n} f\left(a + [k-1]\frac{b-a}{n}\right)\frac{b-a}{n} = \sum_{k=0}^{n-1} f\left(a + k\frac{b-a}{n}\right)\frac{b-a}{n}$

Notice that we may also start at b.

Then $L_n = \sum_{k=1}^{n} f\left(b - k\frac{b-a}{n}\right)\frac{b-a}{n}$.

__Ex.__ Express R_n as a Riemann sum starting from b for the integral $\int_a^b f(x)\,dx$.

__Solution:__ The scheme of construction is



$$x_k = b - k\,\Delta x_n = b - k\frac{b-a}{n} \qquad 0 \leq k \leq n-1$$

Thus $R_n = \sum_{k=0}^{n-1} f\left(b - k\frac{b-a}{n}\right)\frac{b-a}{n}$.

__Ex.__ Express $\int_a^b f(x)\,dx$ as $\lim_{n\to\infty} M_n$

where M_n is $\sum_{k=1}^{n} f(x_k^*)\,\Delta x_n$ where x_k^* is

the central point (midpoint) of the k^{th} interval segment.

__Solution:__

The midpoint of interval k 

is the average of

x_{k-1} and x_k. Thus $x_k^* = \frac{x_{k-1}+x_k}{2}$.

Since $x_k = a + k\frac{b-a}{n}$ and $x_{k-1} = a + (k-1)\frac{b-a}{n}$

$$\frac{x_{k-1}+x_k}{2} = a + \frac{2k-1}{2}\ \frac{b-a}{n}$$

Thus $$M_n = \sum_{k=1}^{n} f\left(a + \frac{2k-1}{2}\ \frac{b-a}{n}\right)\frac{b-a}{n}$$

<u>Ex.</u> Identify the integral $\int_a^b f(x)\,dx$ from the Riemman sum.

(a) $$\lim_{n\to\infty} \sum_{k=1}^{n}\left[\left(3+k\tfrac{2}{n}\right)^2+1\right]\frac{2}{n}$$

(b) $$\lim_{n\to\infty} \sum_{k=1}^{n}\left[\left(-1+k\tfrac{4}{n}\right)^2+5\left(-1+k\tfrac{4}{n}\right)+7\right]\frac{4}{n}$$

(c) $$\lim_{n\to\infty} \sum_{k=0}^{n-1}\left[\sqrt{1+k\tfrac{1}{n}}+e^{(1+k\frac{1}{n})}\right]\frac{1}{n}$$

(d) $$\lim_{n\to\infty}\left(e^{-1+\cancel{k}\frac{1}{n}}\frac{1}{n} + e^{-1+2\frac{1}{n}}\frac{1}{n} + \ldots + e^{-1+n\frac{1}{n}}\frac{1}{n}\right)$$

Solution:

(a) $$\sum_{k=1}^{n} \underbrace{\Big[\big(\underbrace{3+k\tfrac{2}{n}}_{x_k}\big)^2 + 1\Big]}_{f(x_k)} \underbrace{\frac{2}{n}}_{\Delta x_n}$$

so $f(x_k) = x_k^2 + 1$ or $f(x) = x^2+1$

setting $k=0$ we see that $a=3$.

setting $k=n$ we get $b = 3+2 = 5$

Hence $\int_a^b f(x)\,dx = \int_3^5 (x^2+1)\,dx$

(b) $$\sum_{k=1}^{n} \Big[\underbrace{\big(\underbrace{-1+k\tfrac{4}{n}}_{x_k}\big)^2 + 5\big(\underbrace{-1+k\tfrac{4}{n}}_{x_k}\big) + 7}_{f(x_k)}\Big] \underbrace{\frac{4}{n}}_{\Delta x_n}$$

so $f(x) = x^2 + 5x + 7$; $x_k = -1 + k\frac{4}{n}$

setting $k=0$ we get lower bound $a=-1$

setting $k=n$ we get upper bound $b = -1+4 = 3$

Hence $\int_a^b f(x)\,dx = \int_{-1}^{3} (x^2+5x+7)\,dx.$

(c)

$$\sum_{k=0}^{n-1}\underbrace{\left[\sqrt{\underbrace{1+k\tfrac{1}{n}}_{x_k}}+e^{(1+k\frac{1}{n})}\right]}_{f(x_k)}\underbrace{\frac{1}{n}}_{\Delta x_n}$$

so $f(x) = \sqrt{x} + e^x$; $x_k = 1 + k\frac{1}{n}$

setting $k=0$ and $k=n$ gives the bounds.

Hence $\int_a^b f(x)\,dx = \int_1^2 \left(\sqrt{x} + e^x\right) dx$.

(d) This is just the sum

$$\sum_{k=1}^{n} e^{-1+k\frac{1}{n}}\frac{1}{n}$$

clearly $\int_a^b f(x)\,dx = \int_{-1}^{0} e^x\,dx$.